RESEARCH GROUP IN MATHEMATICAL INEQUALITIES AND APPLICATIONS

PROBLEM CORNER

Problem 4, (2009), Solution No. 1

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Solution:

$$tan^2\frac{B}{2}tan^2\frac{C}{2}=\left(\frac{sinB}{1+cosB}\right)\left(\frac{1-cosC}{sinC}\right)=\left(\frac{b}{c}\right)^2\left(\frac{(a-b)^2-c^2}{(a+c)^2-b^2}\right)^2\left(\frac{2ac}{2ab}\right)^2=\left(\frac{a-b-c}{a+b+c}\right)^2.$$

So

$$\begin{aligned}&\frac{a}{b+c}tan^2\frac{B}{2}tan^2\frac{C}{2}+\frac{b}{c+a}tan^2\frac{C}{2}tan^2\frac{A}{2}+\frac{c}{a+b}tan^2\frac{A}{2}tan^2\frac{B}{2}\\=&\frac{a}{b+c}\left(\frac{a-b-c}{a+b+c}\right)^2+\frac{b}{c+a}\left(\frac{b-c-a}{a+b+c}\right)^2+\frac{c}{a+b}\left(\frac{c-a-b}{a+b+c}\right)^2\end{aligned}$$

Let $a=y+z, b=z+x, c=x+y$, then we will have:

$$\frac{y+z}{2x+y+z}\cdot\frac{x^2}{(x+y+z)^2}+\frac{x+z}{x+2y+z}\cdot\frac{y^2}{(x+y+z)^2}+\frac{x+y}{x+y+2z}\cdot\frac{z^2}{(x+y+z)^2}\leq\frac{1}{6}.$$

or it is equivalent to

$$\sum_{sym}x^5+11\sum_{sym}x^4y+23\sum_{sym}x^3y^2+24\sum_{sym}x^3yz+37\sum_{sym}x^2y^2z$$

$$\geq 6\sum_{sym} x^4y + 30\sum_{sym} x^3y^2 + 18\sum_{sym} x^3yz + 42\sum_{sym} x^2y^2z.$$

i.e.

$$\sum_{sym} x^5 + 5\sum_{sym} x^4y + 6\sum_{sym} x^3yz \geq 7\sum_{sym} x^3y^2 + 5\sum_{sym} x^2y^2z.$$

which is right, because

$$\sum_{sym} x^5 + \sum_{sym} x^3yz \geq 2\sum_{sym} x^{\frac{10}{3}}y^{\frac{5}{3}} \geq 2\sum_{sym} x^3y^2$$

and

$$5\sum_{sym} x^4y + 5\sum_{sym} x^3yz \geq 5\sum_{sym} x^3y^2 + 5\sum_{sym} x^2y^2z.$$