RESEARCH GROUP IN MATHEMATICAL INEQUALITIES AND APPLICATIONS

PROBLEM CORNER

Problem 5, (2009), Solution No. 1

Manh Dung Nguyen

Student, Special High School for Gifted Students
Hanoi University of Science
182 Luong The Vinh Street, Thanh Xuan District
Hanoi, Vietnam
Email: nguyendunghus@gmail.com

Received: 28 March, 2009

Problem 5. Let an arbitrary $\triangle ABC$ be in a plane which has circumradius R and inradius r. Denote by $w_a, w_b, w_c; h_a, h_b, h_c$ [†] three internal bisectors and three altitudes from vertex A, B, C respectively. Prove or disprove that

$$\frac{w_a}{h_a} + \frac{w_b}{h_b} + \frac{w_c}{h_c} \leq \sqrt{\frac{4R}{r} + 1}$$

Solution. Denote s is the semi-perimeter of the triangle. Using two known formulas $w_a = \dfrac{2\sqrt{bcs(s-a)}}{b+c}$, $h_a = \dfrac{2\sqrt{s(s-a)(s-b)(s-c)}}{a}$ and the AM-GM Inequality, we have

$$\frac{w_a}{h_a} = \frac{a}{b+c}\sqrt{\frac{bc}{(s-b)(s-c)}} \leq \frac{a}{2\sqrt{(s-b)(s-c)}}$$

RGMIA-pc-5-09-s1

[†] sign of inequality corrected

Similarly, we get

$$\frac{w_a}{h_a}+\frac{w_b}{h_b}+\frac{w_c}{h_c}\le\sum\frac{a}{2\sqrt{(s-b)(s-c)}}$$

Set $a=y+z, b=z+x, c=x+y$ where x,y,z are positive real numbers, we have

$$\sum\frac{a}{2\sqrt{(s-b)(s-c)}}=\sum\frac{y+z}{2\sqrt{yz}}$$

and

$$\sqrt{\frac{4R}{r}+1}=\sqrt{\frac{(x+y)(y+z)(z+x)}{xyz}+1}=\sqrt{\frac{(x+y+z)(xy+yz+zx)}{xyz}}$$

So it suffices to show that

$$\sum\frac{y+z}{2\sqrt{yz}}\le\sqrt{\frac{(x+y+z)(xy+yz+zx)}{xyz}}$$

or

$$\sum\sqrt{x}(y+z)\le 2\sqrt{(x+y+z)(xy+yz+zx)}$$

By the Cauchy-Schwarz Inequality, we have

$$\begin{aligned}\left(\sum\sqrt{x}(y+z)\right)^2&=\left(\sum\sqrt{xy}(\sqrt{x}+\sqrt{y})\right)^2\\&\le(xy+yz+zx)\left(\sum(\sqrt{x}+\sqrt{y})^2\right)\\&\le 2(xy+yz+zx)(x+y+z)\end{aligned}$$

as desired.
Equality holds when $x=y=z$ or $a=b=c$.