Research Group in Mathematical Inequalities and Applications

$v(G) > \sum_{m \in G} v(m)$

The value of the Group is greater than the sum of the values of its members.

Problem Corner

Problem 1, (2010), Solution No. 1

Seiichi Manyama
Graduate School of Science
Osaka University, Japan
Email: `manchanr4@gmail.com`

Received: 19 March, 2010

Solution. We prove the inequality

$$1 - \frac{1}{\ln(2k)} < y_k \tag{1}$$

holds for $k \geq 10$.

From $1 - \frac{1}{\ln(20)} = 0.66\cdots$ and $y_{10} = 0.67\cdots$, the inequality (1) is true for $k = 10$. So we prove the inequality (1) is true for $k \geq 11$.

In order to prove this, we show that

$$f(x) = \frac{\ln(1 + \frac{1}{\ln x})}{x \ln x} > e \quad for \quad 0 < x \leq \frac{1}{22}. \tag{2}$$

We have

$$f'(x) = -\frac{1 + (\ln x + 1)^2 \ln(1 + \frac{1}{\ln x})}{(\ln x + 1)(x \ln x)^2}.$$

Since $\ln(1+t) < t < -(\frac{t}{t+1})^2$ for $-\frac{1}{\ln(22)} \leq t < 0$, we get

$$1 + (\frac{1}{t} + 1)^2 \ln(1+t) < 0 \quad for \quad -\frac{1}{\ln(22)} \leq t < 0.$$

From this, $f'(x) < 0$ for $0 < x \leq \frac{1}{22}$. Since $f(\frac{1}{22}) = 2.78 \cdots > e$, the inequality (2) is true.

Let $g_x(a) = x^{x^a}$ and $h_x(a) = g_x(a) - a$, where $0 < x \leq \frac{1}{22}$ and $-\infty < a < \infty$. The inequality (2) is equivalent to

(3) $$h_x(1 + \frac{1}{\ln x}) > 0.$$

We get

(4) $$h_x(1) = x^x - 1 < 0.$$

We have

$$g_x'(a) = x^a (\ln x)^2 g_x(a) > 0,$$
$$g_x''(a) = x^a (\ln x)^3 (1 + x^a \ln x) g_x(a).$$

Since

$$1 + x^a \ln x \geq 1 + x^{1+\frac{1}{\ln x}} \ln x = 1 + ex \ln x > 0 \quad for \quad 1 + \frac{1}{\ln x} \leq a,$$

$g_x''(a) < 0$ for $1 + \frac{1}{\ln x} \leq a$. Therefore, $h_x''(a) < 0$ for $1 + \frac{1}{\ln x} \leq a$. By the inequality (3) and the inequality (4), there is only one real α_x such that $1 + \frac{1}{\ln x} < \alpha_x < 1$ and $g_x(\alpha_x) = \alpha_x$.

Since $\alpha_{\frac{1}{2k}} < 1$ and $g'_{\frac{1}{2k}}(a) > 0$,

$$(\alpha_{\frac{1}{2k}} =) g_{\frac{1}{2k}}(\alpha_{\frac{1}{2k}}) < g_{\frac{1}{2k}}(1).$$

Since $\alpha_{\frac{1}{2k}} < g_{\frac{1}{2k}}(1)$ and $g'_{\frac{1}{2k}}(a) > 0$,

$$(\alpha_{\frac{1}{2k}} =) g_{\frac{1}{2k}}(\alpha_{\frac{1}{2k}}) < g_{\frac{1}{2k}}(g_{\frac{1}{2k}}(1)).$$

In the same way, we get

$$(\alpha_{\frac{1}{2k}} =) g_{\frac{1}{2k}}(\alpha_{\frac{1}{2k}}) < y_k.$$

Therefore,

$$1 - \frac{1}{\ln(2k)} = 1 + \frac{1}{\ln \frac{1}{2k}} < \alpha_{\frac{1}{2k}} < y_k.$$

□

References

[1] Ovidiu Furdui, Problem 1, (2010), *Research Group In Mathematical Inequalities And Applications*.