Research Group in Mathematical Inequalities and Applications

$v(G) > \sum_{m \in G} v(m)$

The value of the Group is greater than the sum of the values of its members.

Problem Corner

Problem 2, (2010), Solution No. 3

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Solution:

First of all, we note that

(1) $F(x,x) = 0, \forall x \in I.$
(2) $F(x,y) = F(y,x), \forall x, y \in I.$

Since f is convex then there is an increasing function $g : (a,b) \to \mathbb{R}$ and a point $c \in (a,b)$ (see [1], pp 9–10), such that

(1) $$f(t) - f(c) = \int_c^t g(y)\, dy$$

which follows that

$$f(s) - f(c) = \int_c^s g(y)\,dy. \tag{2}$$

Adding (3) and (4), we get

$$f(t) + f(s) - 2f(c) = \int_c^t g(y)\,dy + \int_c^s g(y)\,dy, \tag{3}$$

putting $c = \frac{s+t}{2}$, we get

$$F(s,t) = f(t) + f(s) - 2f\left(\frac{s+t}{2}\right) = \int_{\frac{s+t}{2}}^t g(y)\,dy + \int_{\frac{s+t}{2}}^s g(y)\,dy,$$

Hence,

$$0 \le \sup_{t,s\in I} F(s,t) = \sup_{t,s\in I}\left(\int_{\frac{s+t}{2}}^t g(y)\,dy + \int_{\frac{s+t}{2}}^s g(y)\,dy\right)$$

Since g is increasing, therefore the $\sup$ is satisfied if one choose 's' as far as possible from 't', which is satisfied if $s = a$ (or $s = b$) and $t = b$ (or $t = a$). Thus,

$$\begin{aligned}
\sup_{t,s\in I} F(s,t) &= \int_{\frac{a+b}{2}}^b g(y)\,dy + \int_{\frac{a+b}{2}}^a g(y)\,dy \\
&= f(b) - f\left(\frac{a+b}{2}\right) + f(a) - f\left(\frac{a+b}{2}\right) \\
&= f(a) + f(b) - 2f\left(\frac{a+b}{2}\right) \\
&= F(a,b) = \max_{t,s\in I} F(s,t).
\end{aligned}$$

which completes the proof.

Moreover, let us show that $F(s,t) \ge 0$, for all $s,t \in I$. Since f is convex on I, then $f'_{-}(x)$ and $f'_{+}(x)$ exists and are increasing, so that f has at least one line of support at each $x_0 \in (a,b)$. By choosing $m \in [f'_{-}(x), f'_{+}(x)]$, we have

$$\frac{f(x) - f(x_0)}{x - x_0} \ge (\le)\, m$$

according as $x > x_0$ or $x_0 > x$. In either case, we have

$$f(x) \ge f(x_0) + m(x - x_0).$$

therefore, for all $t, s \in I$ we write

(4) $$f(t) \geq f(x_0) + m(t - x_0).$$

and

(5) $$f(s) \geq f(x_0) + m(s - x_0).$$

Adding (6) and (7) to each other, we get

(6) $$f(t) + f(s) \geq 2f(x_0) + m(t + s - 2x_0),$$

set $x_0 = \frac{s+t}{2}$, we get

(7) $$f(t) + f(s) - 2f\left(\frac{s+t}{2}\right) \geq 0,$$

which shows that $F \geq 0$.

References

[1] A.W. Roberts and D.E. Varberg, Convex functions, New York, Academic Press, 1973.