Research Group in Mathematical Inequalities and Applications

$v(G) > \sum_{m \in G} v(m)$

The value of the Group is greater than the sum of the values of its members.

Problem Corner

Problem 6, (2010), Solution No. 1

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First we show, that the inequalities

(1) $$\frac{1}{2}\int_a^b \frac{f(u)du}{\arcsin\sqrt{\frac{u-a}{b-a}}} \leq \int_a^b \frac{f(u)du}{\sqrt{(u-a)(b-u)}} \leq \frac{\pi}{4}\int_a^b \frac{f(u)du}{\arcsin\sqrt{\frac{u-a}{b-a}}}$$

cannot be true for all a, b and f.
The substitution $v = \frac{u-a}{b-a}$ yields

(2) $$\int_a^b \frac{f(u)du}{\arcsin\sqrt{\frac{u-a}{b-a}}} = (b-a)\int_0^1 \frac{f(a+v(b-a))dv}{\arcsin\sqrt{v}}$$

(3) $$\int_a^b \frac{f(u)du}{\sqrt{(u-a)(b-u)}} = \int_0^1 \frac{f(a+v(b-a))dv}{\sqrt{v(1-v)}}.$$

Taking $f(x) \equiv 1$ inequality (1) reads

$$\frac{b-a}{2}\int_0^1 \frac{dv}{\arcsin\sqrt{v}} \le \int_0^1 \frac{dv}{\sqrt{v(1-v)}} \le \frac{\pi(b-a)}{4}\int_0^1 \frac{dv}{\arcsin\sqrt{v}}$$

and we see that appropriate choice of a and b can destroy both inequalities.

Let us try to save partially the problem by showing the following

Theorem 1. *If f is continuous and nonnegative, then for $a < b$*

$$\frac{1}{b-a}\int_a^b \frac{f(u)du}{\arcsin\sqrt{\frac{u-a}{b-a}}} \le \int_a^b \frac{f(u)du}{\sqrt{(u-a)(b-u)}}$$

and there is no K such that

$$\int_a^b \frac{f(u)du}{\sqrt{(u-a)(b-u)}} \le \frac{K}{b-a}\int_a^b \frac{f(u)du}{\arcsin\sqrt{\frac{u-a}{b-a}}}$$

holds.

Proof. Using (2) and (3) our inequality can be rewritten as

$$\int_0^1 \frac{f(a+v(b-a))dv}{\arcsin\sqrt{v}} \le \int_0^1 \frac{f(a+v(b-a))dv}{\sqrt{v(1-v)}}$$

and to prove it, it is enough to show that

$$\frac{1}{\arcsin\sqrt{v}} \le \frac{1}{\sqrt{v(1-v)}}.$$

The function $\arcsin t$ is convex, therefore $\frac{\arcsin t}{t}$ increases for $t > 0$, and so does $\frac{\arcsin\sqrt{v}}{\sqrt{v}}$. The function $\frac{1}{\sqrt{1-v}}$ is increasing also, thus their product monotonically grows from $\lim_{v\to 0}\frac{\arcsin\sqrt{v}}{\sqrt{v(1-v)}} = 1$ to $\lim_{v\to 1}\frac{\arcsin\sqrt{v}}{\sqrt{v(1-v)}} = \infty$. This completes the first part of our proof.

To prove the second part, assume for simplicity that $(a,b) = (0,1)$. For arbitrary $\alpha > 1$ there exists $0 < v_\alpha < 1$ such that

$$\frac{\alpha}{\arcsin\sqrt{v}} \le \frac{1}{\sqrt{v(1-v)}}$$

holds for all $v > v_\alpha$. Take a function f which equals zero in $[0, v_\alpha]$. Then

$$\int_0^1 \frac{f(v)dv}{\sqrt{v(1-v)}} = \int_{v_\alpha}^1 \frac{f(v)dv}{\sqrt{v(1-v)}} \ge \alpha\int_{v_\alpha}^1 \frac{f(v)dv}{\arcsin\sqrt{v}} = \alpha\int_0^1 \frac{f(v)dv}{\arcsin\sqrt{v}}.$$

□